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# BOUNDARY LAYER PROBLEM FOR THE SYSTEM OF THE FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS BY GENERAL NONLOCAL BOUNDARY CONDITIONS

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**Abstract.** In this paper we consider a boundary layer problem (singular perturbation problem) which consists of the first order system of differential equations. For the given problem, we determine if the boundary layers exist or not. Then using the necessary conditions and fundamental solution of the given system of differential equations, we obtain some sufficient conditions for the absence of the boundary layer.

***Keywords:*** *singular perturbation problems, necessary conditions, nonlocal boundary conditions, boundary layer.*

***AMS Subject Classification:*** *76D10, 76F40.*

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## 1. Introduction

One of the important subjects in applied mathematics is the theory of singular perturbation problems. The mathematical model for this kind of problem usually is in the form of either ordinary differential equations (O.D.E) or partial differential equations (PDE) in which the highest derivative is multiplied by some powers of $\varepsilon$ as a positive small parameter [17], [18].The purpose of the theory of singular perturbations is to solve a differential equation with some initial or boundary conditions with small parameter $\varepsilon$. If the solution of the differential equation $\varepsilon y_\varepsilon^{(n)} + f(y_\varepsilon^{(n-1)}, y_\varepsilon^{(n-2)},..., y'_\varepsilon, y, x) = 0$, (when $\varepsilon$ is chosen to be zero) is the same solution as the limit of the solution when $\varepsilon \downarrow 0$, then we say our problem has no a boundary layer. In other words, the limit of the solution i.e., $\lim_{\varepsilon \downarrow 0} y(x,\varepsilon) = y_0(x)$, satisfies the given boundary conditions. Otherwise we said that the boundary layer exists, [7,8, 10, 16]. Naturally the first case is more desirable than the other ones [14,15]. On the other hand, since the structure of the approximate solutions of these problems depends on the place of the boundary layer, the determination of the points where the boundary layer is forming is important. For this we use some compatibility conditions which we call the necessary conditions. For the first time these conditions have been applied by

A.V. Bitsadze [23]. Then N. Aliev , M. Jahanshahi, SH. Rezapour and S. M. Hosseini [2-4,12,13] used these conditions for the determining of well-posedness of the boundary value problems for PDE's. On the other hand, Bhupesh K. Tripathi [23] considered singular second order boundary value problem with singularities in the coefficients of the equation. For these problems they constructed asymptotic approximate solutions. Also Ghazala Akram and Hamood ur Rehman considered a boundary layer problem of the fourth order ODE whit local boundary conditions [1]. According to these facts, we decided to apply these conditions for the determining the boundary layers for the singular perturbation problems. In some works of M. Jahanshahi & A. R. Sarakhsi [11,20-22] and S.Ashrafi and N.Aliev [5,6], presented sufficient conditions for some singular perturbation problems for ODEs which do not have boundary layers.

In [20] the authors have studied the following perturbation problem with general non-local

$$l_\varepsilon y_\varepsilon(x) \equiv -\varepsilon y_\varepsilon^{(iv)}(x) + a y_\varepsilon^{''}(x) + b y_\varepsilon(x) = 0, x \in (0,1)$$

$$l_i y_\varepsilon(x) \equiv \sum_{j=0}^{3} [\alpha_{ij}^{(0)} y_\varepsilon^{(j)}(0) + \alpha_{ij}^{(1)} y_\varepsilon^{(j)}(1)] = \alpha_i,\ i = 1,2,3,4.$$

Also for the second order ODE with local boundary conditions see [21].

## 2. Mathematical statement of problem

In this paper we investigate the following singular perturbation boundary value problem,

$$l y_\varepsilon \equiv \varepsilon \mathrm{a} y_\varepsilon'(x) + b y_\varepsilon(x) = f(x), \qquad x \in (0,1), \tag{1}$$

$$\alpha y_\varepsilon(0) + \beta y_\varepsilon(1) = 0, \tag{2}$$

where $\mathrm{a}, b, \alpha$ and $\beta$ are *nxn* matrices, $\varepsilon \geq 0$ is a small parametr, $f(x)$ is a known continous vector function, $y_\varepsilon(x)$ is n-dimensional unknown vector function.

We also assume that boudary conditions (2) are independent and

$$\det \mathrm{a} = |\mathrm{a}| \neq 0. \tag{3}$$

As mentioned in the introduction we derive some sufficient conditions for the problem's data, so that the problem has no a boundary layer. To do this, we use the generalized solution (or fundamental solution) of adjoint equation of (1) to obtain some necessary conditions for the arbitrary solution of (1). Then, by matching boundary conditions and obtained necessary conditions we will give some sufficient conditions under which the given boundary layer problem have no boundary layer in the boundary points.

We reduce equation (1) to the algebraic system when $\varepsilon \to 0$. This algebraic system may not satisfy boundary conditions (2). In this case we have a boundary layer. The main goal of this paper is presentation some sufficient conditions under which we have no a boundary layer.

Considering condition (3) and multipling bothsides of (1) by a-1 yeilds

$$\varepsilon y_\varepsilon'(x) + \mathrm{a}^{-1} b y_\varepsilon(x) = \mathrm{a}^{-1} f(x), \tag{4}$$

Without lose of generality , we assume

$$a = I \,, \tag{5}$$

Hence we can consider the following system instead of (10):

$$\varepsilon y'_{\varepsilon}(x) + b y_{\varepsilon}(x) = f(x) \,, \tag{6}$$

Make the subsitution

$$y_{\varepsilon}(x) = T u_{\varepsilon}(x) \,.$$

From linear algebra, there is an invertible matrix T for which we have

$$T^{-1} b T = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ 0 & 0 & \lambda_n \end{pmatrix} , \tag{7}$$

where $\lambda_k$ - ( $k = \overline{1,\, n}$ ), are the eigenvalues of the matrix b. Assme that:

$$\lambda_k \neq 0 \,,\; k = \overline{1,\, n} \,; \qquad \lambda_k < 0 \,,\; k = \overline{1,\, n_1} \,; \qquad \lambda_k > 0 \,,\; k = \overline{n_1 + 1,\, n} \,, \tag{8}$$

Since (7) is a diagonal matrix. If we use the following change of the variable

$$y_{\varepsilon}(x) = T u_{\varepsilon}(x) \tag{9}$$

then from (6) we have:

$$\varepsilon T u'_{\varepsilon}(x) + b T u_{\varepsilon}(x) = f(x) \,,$$

or

$$\varepsilon u'_{\varepsilon}(x) + \Lambda u_{\varepsilon}(x) = T^{-1} f(x) \,.$$

Finally without lose of generality, we will consider the following system instead of system (6).

$$\varepsilon y'_{i\varepsilon}(x) + \lambda_i y_{i\varepsilon}(x) = T^{-1} f(x) \,, \quad x \in (0,\, 1) \,,\; i = \overline{1,\, n} \,, \tag{10}$$

## 3. Adjoint system and its fundamental solution

Now calculate the adjoint system and its fundamental solution (generalized solution) for system (10). For this purpose, multipling bothsides of (10) by $z_{i\varepsilon}(x)$ and integrating over the interval (0, 1) implies

$$\varepsilon \int_0^1 z_{i\varepsilon}(x) y'_{i\varepsilon}(x) dx + \lambda_i \int_0^1 z_{i\varepsilon}(x) y_{i\varepsilon}(x) dx = \int_0^1 z_{i\varepsilon}(x) f_i(x) dx \,, \quad i = \overline{1,\, n} \,.$$

Integrating by parts the first integral term of theabove relation we get

$$\varepsilon z_{i\varepsilon}(x) y_{i\varepsilon}(x) \Big|_{x=0}^{1} + \int_0^1 \left[ -\varepsilon z'_{i\varepsilon}(x) + \lambda_i z_{i\varepsilon}(x) \right] y_{i\varepsilon}(x) dx = \int_0^1 z_{i\varepsilon}(x) f_i(x) dx \tag{11}$$

Hence for the adjoint equation of (10) we have

$$l_i^* z_{i\varepsilon} \equiv -\varepsilon z'_{i\varepsilon}(x) + \lambda_i z_{i\varepsilon}(x) = g_i(x) \,,\; i = \overline{1,\, n} \,, \tag{12}$$

where $g_i(x)$ $i = \overline{1,\, n}$ are arbitrary continous functions.

Now we obtain the fundamental solution of (12). For this, both sides of (12) is divieded by $(-\varepsilon)$

$$z'_{i\varepsilon}(x) - \frac{\lambda_i}{\varepsilon} z_{i\varepsilon}(x) = -\frac{1}{\varepsilon} g_i(x).$$

A particular solution is

$$z_{i\varepsilon}(x) = -\frac{1}{\varepsilon}\int_0^x e^{\frac{\lambda_i}{\varepsilon}(x-\xi)} g_i(\xi)d\xi, \quad i = \overline{1, n_1}; \tag{12$_1$}$$

or

$$z_{i\varepsilon}(x) = -\frac{1}{\varepsilon}\int_1^x e^{\frac{\lambda_i}{\varepsilon}(x-\xi)} g_i(\xi)d\xi, \quad i = \overline{n_1+1, n}; \tag{12$_2$}$$

Therefore the fundamental solution of (12) will be

$$Z_{i\varepsilon}(x-\xi) = -\frac{\theta(x-\xi)}{\varepsilon} e^{\frac{\lambda_i}{\varepsilon}(x-\xi)}, \quad i = \overline{1, n_1}; \tag{13$_1$}$$

and

$$Z_{i\varepsilon}(x-\xi) = \frac{\theta(\xi-x)}{\varepsilon} e^{-\frac{\lambda_i}{\varepsilon}(\xi-x)}, \quad i = \overline{n_1+1, n}; \tag{13$_2$}$$

where $\theta(x-\xi)$ is the unit Heaviside function.

## 4. Necessary conditions

We try to find arbitrary conditions for the solution of (6) in the given interval [0,1]. We search the necessary conditions which the solution itself satisfy. To do this, we multiple both sides of (6) by $z_{i\varepsilon}(x)$ and then integrate over [0,1]

$$\varepsilon\int_0^1 y'_{i\varepsilon}(x)Z_{i\varepsilon}(x-\xi)dx + \lambda_i\int_0^1 y_{i\varepsilon}(x)Z_{i\varepsilon}(x-\xi)dx = \int_0^1 Z_{i\varepsilon}(x-\xi) f_i(x)dx,$$

or

$$\varepsilon y_{i\varepsilon}(x)Z_{i\varepsilon}(x-\xi)\Big|_{x=0}^1 + \int_0^1 \left[-\varepsilon Z'_{i\varepsilon}(x-\xi) + \lambda_i Z_{i\varepsilon}(x-\xi)\right] y_{i\varepsilon}(x)dx =$$

$$= \int_0^1 Z_{i\varepsilon}(x-\xi) f_i(x)dx,$$

Now by considering fundamental solution and the property of Dirac's delta function we have

$$\varepsilon Z_{i\varepsilon}(-\xi)y_{i\varepsilon}(0) - \varepsilon Z_{i\varepsilon}(1-\xi)y_{i\varepsilon}(1) + \int_0^1 Z_{i\varepsilon}(x-\xi) f_i(x)dx =$$

$$= \begin{cases} y_{i\varepsilon}(\xi), & \xi \in (0,1), \\ \frac{1}{2} y_{i\varepsilon}, & \xi = 0,\ \xi = 1, \end{cases} \qquad i = \overline{1, n}. \tag{14}$$

The first case of (14) gives an analytic solution for equation (14) and the second case gives the desirable necessary conditions.

These ralations can be written in the following simple forms:

$$\frac{1}{2}y_{i\varepsilon}(0)=-\frac{1}{2}y_{i\varepsilon}(0)+e^{\frac{\lambda_i}{\varepsilon}}y_{i\varepsilon}(1)-\frac{1}{\varepsilon}\int_0^1 e^{\frac{\lambda_i}{\varepsilon}x}f_i(x)dx,\quad i=\overline{1,\,n_1},$$
$$\frac{1}{2}y_{i\varepsilon}(1)=e^{-\frac{\lambda_i}{\varepsilon}}y_{i\varepsilon}(0)-\frac{1}{2}y_{i\varepsilon}(1)+\frac{1}{\varepsilon}\int_0^1 e^{-\frac{\lambda_i}{\varepsilon}(1-x)}f_i(x)dx,\quad i=\overline{n_1+1,\,n}.$$

To sum up we conclude the following theorem:

**Theorem 1.** Let $\varepsilon \geq 0$ be a small parametr, $\lambda_i$, i=1,...n , according to the (8) are the strict eigenvalues of the matrix b and $f_i(x)$ are known continous functions. Then the arbitrary solution of system (10) satisfies in the following necessary conditions

$$y_{i\varepsilon}(0)=e^{\frac{\lambda_i}{\varepsilon}}y_{i\varepsilon}(1)-\frac{1}{\varepsilon}\int_0^1 e^{\frac{\lambda_i}{\varepsilon}x}f_i(x)dx,\quad i=\overline{1,\,n_1}\quad;$$
$$y_{i\varepsilon}(1)=e^{-\frac{\lambda_i}{\varepsilon}}y_{i\varepsilon}(0)+\frac{1}{\varepsilon}\int_0^1 e^{-\frac{\lambda_i}{\varepsilon}(1-x)}f_i(x)dx,\quad i=\overline{n_1+1,\,n}\quad;$$

**Remark 1.** In the above relations there is no any unbounded terms when $\varepsilon \to 0$.

## 5. Localization of the boundary conditions

In this section, we reduce nonlocal boundary conditions to loal case. To do this, we rewrite the boundary conditions in the following form

$$\sum_{j=1}^{n}\alpha_{ij}y_{j\varepsilon}(0)+\sum_{j=1}^{n}\beta_{ij}y_{j\varepsilon}(1)=0,\quad i=\overline{1,\,n},$$

or

$$\sum_{j=1}^{n_1}\alpha_{ij}y_{j\varepsilon}(0)+\sum_{j=n_1+1}^{n}\alpha_{ij}y_{j\varepsilon}(0)+\sum_{j=1}^{n_1}\beta_{ij}y_{j\varepsilon}(1)+\sum_{j=n_1+1}^{n}\beta_{ij}y_{j\varepsilon}(1)=0,\quad i=\overline{1,\,n}.$$

According to the obtained necessary conditions in Teorem 1, we have

$$\sum_{j=1}^{n_1}\alpha_{ij}\left[e^{\frac{\lambda_j}{\varepsilon}}y_{j\varepsilon}(1)-\frac{1}{\varepsilon}\int_0^1 e^{\frac{\lambda_j}{\varepsilon}x}f_j(x)dx\right]+\sum_{j=n_1+1}^{n}\alpha_{ij}y_{j\varepsilon}(0)+$$
$$+\sum_{j=1}^{n_1}\beta_{ij}y_{j\varepsilon}(1)+\sum_{j=n_1+1}^{n}\beta_{ij}\left[e^{-\frac{\lambda_j}{\varepsilon}}y_{j\varepsilon}(0)+\frac{1}{\varepsilon}\int_0^1 e^{-\frac{\lambda_j}{\varepsilon}(1-x)}f_j(x)dx\right]=0,\ i=1,n.$$

This algeraic system is rewritten explicity with respect to its unknowns

$$\sum_{j=n_1+1}^{n}\left(\alpha_{ij}+\beta_{ij}e^{-\frac{\lambda_j}{\varepsilon}}\right)y_{j\varepsilon}(0)+\sum_{j=1}^{n_1}\left(\beta_{ij}+\alpha_{ij}e^{\frac{\lambda_j}{\varepsilon}}\right)y_{j\varepsilon}(1)=$$
$$=\sum_{j=1}^{n_1}\frac{\alpha_{ij}}{\varepsilon}\int_0^1 e^{\frac{\lambda_j}{\varepsilon}x}f_j(x)dx-\sum_{j=n_1+1}^{n}\frac{\beta_{ij}}{\varepsilon}\int_0^1 e^{-\frac{\lambda_j}{\varepsilon}(1-x)}f_j(x)dx,\ i=\overline{1,n}\,. \tag{15}$$

Assume the following condition holds

$$\Delta(\varepsilon)=\begin{vmatrix}\Delta_{11} & \Delta_{12} & \dots\dots & \Delta_{1n}\\ \Delta_{21} & \Delta_{22} & \dots\dots & \Delta_{2n}\\ \dots\dots & \dots\dots & \dots\dots & \dots\dots\\ \dots\dots & \dots\dots & \dots\dots & \dots\dots\\ \Delta_{n1} & \Delta_{n2} & \dots\dots & \Delta_{nn}\end{vmatrix}\neq 0, \tag{16}$$

where

$$\Delta_{ij}(\varepsilon)=\beta_{ij}+\alpha_{ij}e^{\frac{\lambda_j}{\varepsilon}}, \qquad i=\overline{1,\ n};\quad j=\overline{1,\ n_1}, \tag{16$_1$}$$

$$\Delta_{ij}(\varepsilon)=\alpha_{ij}+\beta_{ij}e^{-\frac{\lambda_j}{\varepsilon}}, \qquad i=\overline{1,\ n};\quad j=\overline{n_1+1,\ n}, \tag{16$_2$}$$

Then we obtain from (15)

$$\begin{aligned} y_{j\varepsilon}(1)&=\frac{1}{\Delta(\varepsilon)}\sum_{i=1}^{n}\sum_{s=1}^{n_1}\frac{\alpha_{is}\Delta^{ij}(\varepsilon)}{\varepsilon}\int_0^1 e^{\frac{\lambda_s}{\varepsilon}x}f_s(x)dx-\\ &-\frac{1}{\Delta(\varepsilon)}\sum_{i=1}^{n}\sum_{s=n_1+1}^{n}\frac{\beta_{is}\Delta^{ij}(\varepsilon)}{\varepsilon}\int_0^1 e^{-\frac{\lambda_s}{\varepsilon}(1-x)}f_s(x)dx,\quad j=\overline{1,n_1}; \end{aligned} \tag{17$_1$}$$

$$\begin{aligned} y_{j\varepsilon}(0)&=\frac{1}{\Delta(\varepsilon)}\sum_{i=1}^{n}\sum_{s=1}^{n_1}\frac{\alpha_{is}\Delta^{ij}(\varepsilon)}{\varepsilon}\int_0^1 e^{\frac{\lambda_s}{\varepsilon}x}f_s(x)dx-\\ &-\frac{1}{\Delta(\varepsilon)}\sum_{i=1}^{n}\sum_{s=n_1+1}^{n}\frac{\beta_{is}\Delta^{ij}(\varepsilon)}{\varepsilon}\int_0^1 e^{-\frac{\lambda_s}{\varepsilon}(1-x)}f_s(x)dx,\quad j=\overline{n_1+1,n}; \end{aligned} \tag{17$_2$}$$

where $\Delta^{ij}$ is a coefficents of (16) corresponding the element $a_{ij}$.

Therefore we have the follwing theorem.

**Theorem 2.** Under conditions of Theorem 1, when the matrices $\alpha$ and $\beta$ are $n$-order with constant real nubers, and boundary condition (2) are linearly independent, the condition (16) hold, according to relations ($17_1$), ($17_2$) non-local boundary conditions (2) are reduced to the following local boundary conditions

$$y_{j\varepsilon}(0)=e^{\frac{\lambda_j}{\varepsilon}}y_{j\varepsilon}(1)-\frac{1}{\varepsilon}\int_0^1 e^{\frac{\lambda_j}{\varepsilon}x}f_j(x)dx, \qquad j=\overline{1,\ n_1},$$

$$y_{j\varepsilon}(1)=e^{-\frac{\lambda_j}{\varepsilon}}y_{j\varepsilon}(0)+\frac{1}{\varepsilon}\int_0^1 e^{-\frac{\lambda_j}{\varepsilon}(1-x)}f_j(x)dx, \qquad j=\overline{n_1+1,\ n},$$

## 6. Analytic solution

Using the first case of (14), we can write the analytis solution of the main problem as

$$y_{i\varepsilon}(\xi)=e^{\frac{\lambda_i}{\varepsilon}(1-\xi)}y_{i\varepsilon}(1)-\int_0^1\frac{\theta(x-\xi)}{\varepsilon}e^{\frac{\lambda_i}{\varepsilon}(x-\xi)}f_i(x)dx, \qquad i=\overline{1,\ n_1} \tag{18$_1$}$$

$$y_{i\varepsilon}(\xi)=e^{-\frac{\lambda_i}{\varepsilon}\xi}y_{i\varepsilon}(0)+\int_0^1\frac{\theta(\xi-x)}{\varepsilon}e^{-\frac{\lambda_i}{\varepsilon}(\xi-x)}f_i(x)dx, \qquad i=\overline{n_1+1,\ n}. \tag{18$_2$}$$

Now when $\varepsilon \to 0$, we can consider the limit behavior of the solution,

$$y_{i0}(\xi) = -\lim_{\varepsilon\to 0}\int_0^1 \frac{\theta(x-\xi)}{\varepsilon} e^{\frac{\lambda_i}{\varepsilon}(x-\xi)} f_i(x)dx, \qquad i=\overline{1,\ n_1}\ , \tag{$18_{1,0}$}$$

$$y_{i0}(\xi) = \lim_{\varepsilon\to 0}\int_0^1 \frac{\theta(\xi-x)}{\varepsilon} e^{-\frac{\lambda_i}{\varepsilon}(\xi-x)} f_i(x)dx, \qquad i=\overline{n_1+1,\ n}\ . \tag{$18_{2,0}$}$$

If we let

$$\rho = \frac{1}{\varepsilon}, \tag{19}$$

when $\varepsilon \to 0$ then $\rho \to +\infty$, and hence

$$\begin{aligned}\lim_{\rho\to\infty} \rho e^{\rho\lambda_i(x-\xi)} &= \lim_{\rho\to\infty}\frac{\rho}{e^{-\rho\lambda_i(x-\xi)}} = \lim_{\rho\to\infty}\frac{1}{-\lambda_i(x-\xi)e^{-\rho\lambda_i(x-\xi)}} = \\ &= -\frac{1}{\lambda_i(x-\xi)}\lim_{\rho\to\infty} e^{\rho\lambda_i(x-\xi)}, \qquad i=\overline{1,n_1}.\end{aligned} \tag{$20_1$}$$

Since $\lambda_i < 0$ and $x \neq \xi$, when $\rho \to \infty$ then we have

$$\lim_{\rho\to\infty} \rho e^{\rho\lambda_i(x-\xi)} = \frac{1}{|\lambda_i|(x-\xi)}\cdot 0 = 0, \qquad x-\xi \geq 0,\quad i=\overline{1,n_1}, \tag{$20_2$}$$

and if $x=\xi$, then

$$\lim_{\rho\to\infty} \rho e^{\rho\lambda_i(x-\xi)} = \infty. \tag{$20_3$}$$

Finally from ($20_1$) and the integrand of ($18_{1,0}$), according to (10) we have

$$\begin{aligned}y_{i0}(\xi) &= -\int_0^1 \lim_{\varepsilon\to 0}\frac{\theta(x-\xi)}{\varepsilon} e^{\frac{\lambda_i}{\varepsilon}(x-\xi)} f_i(x)dx = \\ &= -\int_0^1 \lim_{\rho\to\infty} \rho e^{\rho\lambda_i(x-\xi)}\theta(x-\xi) f_i(x)dx = \int_0^1 \frac{\theta(x-\xi)}{\lambda_i(x-\xi)}\lim_{\rho\to\infty} e^{\rho\lambda_i(x-\xi)} f_i(x)dx = \\ &= \frac{1}{\lambda_i}\int_0^1 \delta(x-\xi) f_i(x)dx = \frac{f_i(\xi)}{\lambda_i}, \qquad i=\overline{1,\ n_1}.\end{aligned} \tag{21}$$

Threfore we obtain the following sequence which is similar to Dirac's delta function

$$\lim_{\rho\to\infty}\frac{\theta(x-\xi)}{x-\xi} e^{\rho\lambda_i(x-\xi)} = \delta(x-\xi),\quad i=\overline{1,\ n_1}. \tag{22}$$

With same process we have from ($20_3$)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\theta(\xi-x)}{\varepsilon} e^{-\frac{\lambda_i}{\varepsilon}(\xi-x)} = \lim_{\rho\to\infty}\theta(\xi-x)\rho e^{-\rho\lambda_i(\xi-x)} = \\ &\theta(\xi-x)\lim_{\rho\to\infty}\frac{\rho}{e^{\rho\lambda_i(\xi-x)}} = \theta(\xi-x)\lim_{\rho\to\infty}\frac{e^{-\rho\lambda_i(\xi-x)}}{\lambda_i(\xi-x)},\quad i=\overline{n_1+1,\ n}.\end{aligned} \tag{$23_1$}$$

Considering ($23_1$) generally is $x \neq \xi$. Therefore when $x=\xi$, then integrand ($18_{2,0}$) tends to infinity. Also from (10) we have:

$$\lim_{\rho\to\infty}\frac{\theta(x-\xi)}{\lambda_i(\xi-x)}e^{-\rho\lambda_i(\xi-x)}=\frac{1}{\lambda_i}\delta(\xi-x),\quad i=\overline{n_1+1,\ n}. \tag{24}$$

Similarly from (21) and ($18_{2,0}$) we have

$$y_{i0}(\xi)=\frac{f_i(\xi)}{\lambda_i},\qquad i=\overline{n_1+1,\,n}. \tag{25}$$

Therefore we can conclude the following theorem.

**Theorem 3.** Under assumptions of Teorem 1, the limit position of solution of equation (10) is given by (21) and (25).

## 7. Limit behavior of the local boundary conditions

To study this, at first we calculate the limit of relations ($17_1$) and ($17_2$) :

$$y_{j0}(1)=\frac{1}{\Delta(0)}\sum_{i=1}^{n}\sum_{s=1}^{n_1}\alpha_{is}\Delta^{ij}(0)\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{-\rho\lambda_s x}}f_s(x)dx-$$

$$-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\sum_{s=n_1+1}^{n}\beta_{is}\Delta^{ij}(0)\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{\rho\lambda_s(1-x)}}f_s(x)dx=$$

$$=\frac{1}{\Delta(0)}\sum_{i=1}^{n}\sum_{s=1}^{n_1}\alpha_{is}\Delta^{ij}(0)\int_0^1\lim_{\rho\to\infty}\frac{e^{\rho\lambda_s x}}{-\lambda_s x}f_s(x)dx-$$

$$-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\sum_{s=n_1+1}^{n}\beta_{is}\Delta^{ij}(0)\int_0^1\lim_{\rho\to\infty}\frac{e^{-\rho\lambda_s(1-x)}}{\lambda_s(1-x)}f_s(x)dx=$$

$$=-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\frac{f_s(0)}{\lambda_s}-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\frac{f_s(1)}{\lambda_s},\ \ j=\overline{1,\,n_1}. \tag{$26_1$}$$

Similarly for relaion ($17_2$) we have

$$y_{j0}(0)=\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{-\rho\lambda_s x}}f_s(x)dx-$$

$$-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{\rho\lambda_s(1-x)}}f_s(x)dx=$$

$$=\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\int_0^1\lim_{\rho\to\infty}\frac{e^{\rho\lambda_s x}}{-\lambda_s x}f_s(x)dx-$$

$$-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\int_0^1\lim_{\rho\to\infty}\frac{e^{-\rho\lambda_s(1-x)}}{\lambda_s(1-x)}f_s(x)dx=$$

$$=-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\frac{f_s(0)}{\lambda_s}-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\frac{f_s(1)}{\lambda_s},\ \ j=\overline{n_1+1,\,n}. \tag{$26_2$}$$

$$y_{j0}(0)=-\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{-\rho\lambda_j x}}f_j(x)dx=\frac{f_j(0)}{\lambda_j},\qquad j=\overline{1,\,n_1}, \tag{$26_3$}$$

$$y_{j0}(1)=\int_0^1\lim_{\rho\to\infty}\frac{\rho}{e^{\rho\lambda_j(1-x)}}f_j(x)dx=\frac{f_j(1)}{\lambda_j},\qquad j=\overline{n_1+1,\,n}. \tag{$26_4$}$$

Finally we have the following theorem.

**Theorem 4.** Under assumptions of Theorem 2, the limit positions of local boundary conditions will be given by ($26_1$)-($26_4$).

## 8. Place of boundary

We considered the limit position of solution of system (10) as follows

$$y_{i0}(\xi)=\frac{f_i(\xi)}{\lambda_i}, \qquad i=\overline{1,\, n}, \tag{27}$$

These relations include boundary vaues and the other hand, they are the limit positions of solution of main problem (10), (2). Relations (27) satisfy to relations ($26_3$) and ($26_4$).

According the above relations, if the following relations are valid

$$\frac{f_j(1)}{\lambda_j}=-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\frac{f_s(0)}{\lambda_s}-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\frac{f_s(1)}{\lambda_s},\;\; j=\overline{1,\, n_1}, \tag{28}$$

$$\frac{f_j(0)}{\lambda_j}=-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=1}^{n_1}\alpha_{is}\frac{f_s(0)}{\lambda_s}-\frac{1}{\Delta(0)}\sum_{i=1}^{n}\Delta^{ij}(0)\sum_{s=n_1+1}^{n}\beta_{is}\frac{f_s(1)}{\lambda_s},\;\; j=\overline{n_1+1,\, n}, \tag{29}$$

then the boundary conditions are satisfied and we have no boundary layer.

To sum up we have the final theorem.

**Theorem 5.** Under the conditions of Theorem 2 , if at least one of relations (28) is not hold, then a boundary layer is formed at the boundary point $x=1$. Also, if at least one of relations (29) is not hold, then a boundary layer is formed at the boundary point $x=0$. If at least one of relations (28) and (29) is not hold, then boundary layers are formed at the both of the boundary points.

Finally, if both relations (28) and (29) are hold, then problem (1)-(2) has no a boundary layer.

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